CMPT 441-711: Homework 3

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Question 11.2

We modify the HMM by splitting states F (for fair coin) and L (for loaded coin) into ten states $F_1 \dots F_{10}$ and $L_1 \dots L_{10}$ respectively. Each of these states will have the same emit probabilities as the state F, L. All incoming edges in HMM which were pointing to the F will now point to the F_1, and all outgoing edges (including self-loops) from F will become outgoing edges of F_{10} (all those edges will retain same probability). For each $F_i, i < 10$ we'll add only one outgoing edge to F_{i+1} with probability 1 (all other outgoing edges from F_i will have probability 0). The same rule applies for L states. This way, we force the HMM to stay in state L without visiting state F at least 10 times. Now, we can apply Viterbi's algorithm on this HMM without any modifications.

Question 11.4

We will simply run Viretbi's algorithm for the given HMM, with the given sequence. We'll just add start state S which does not emit anything (i.e. emission probabilities are zero), and which points to states α and β with probability 0.5. Let $S_{s,i}$ be the maximum probability for emission of i characters that ends up in the state s. $S_{s,i}$ will be obtained by the Viterbi's log recurrence:

$$S_{s,i} = \log e_s(x_i) + \max_{t \in \{S,\alpha,\beta\}} \{S_{t,i-1} + \log a_{ts}\}.$$

We will also keep matrix $p_{s,i}$ as a state matrix, where

$$p_{s,i} = \text{argmax}\{S_{t,i-1} + \log a_{ts}\}.$$

After running the Viterbi's algorithm for start state α, we'll obtain following matrices (we'll use base 2 logarithm):

$S_{s,i}$		G	G	C	T
S	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$
α	$-\infty$	-2.32193	-3.79586	-7.26979	-10.7437
β	$-\infty$	-3.32193	-5.79586	-7.68483	-9.5738

$p_{s,i}$	G	G	C	T
α	S	α	α	α
β	S	β	β	β

As $S_{\beta,4} > S_{\alpha,4}$, best possible path in HMM is $S \to \beta \to \beta \to \beta \to \beta$ with probability

$$2 \times 2^{-9.5738} = 2 \times \frac{6561}{5000000}$$

(we multiply result with 2 because we included 1/2 probability of arriving from state S to state β)

Question 11.5

As each coin toss is independent, probability of some sequence is simply the product of probabilities for each particular outcome. We need to compare two probabilities: (i) if we start with fair coin, and (ii) if we

start with loaded coin. In out sequence we have 10 tails and 7 heads. Thus, if we choose fair coin, sequence probability is

$$P_F = \left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{7} = \frac{1}{2^{17}} \Rightarrow \log P_F = -17.$$

In the same way we have

$$P_H = \left(\frac{3}{4}\right)^{10}\left(\frac{1}{4}\right)^{7} = \frac{3^{10}}{4^{17}} \Rightarrow \log P_H = 10\log 3 - 17\log 2^2 = 10\log 3 - 34 = 15.85 - 34 = -18.15.$$

Since $P_F > P_L$, thus the dealer has chosen the fair most probably.

Question 11.6

- Our alphabet is $\Sigma = \{1, 2, 3\}$. States are $Q = \{S, D_1, D_2, E\}$ (S for start and E for end state). Transition and emission probabilities are given in the table below:

$a_{s,t}$	S	D_1	D_2	E
S	0	1/2	1/2	0
D_1	0	1/2	1/4	1/4
D_2	0	1/4	1/2	1/4
E	0	0	0	0

$e_s(\cdot)$	S	D_1	D_2	E
1	0	1/2	1/4	0
2	0	1/4	1/2	0
3	0	1/4	1/4	0

- We'll use Viterbi's algorithm with backtracking matrix, as in previous problems. After running the Viterbi's algorithm our matrices are:

$a_{s,t}$		1	1	2	1	2	2
S	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
D_1	$-\infty$	-2	-4	-7	-9	-12	-15
D_2	$-\infty$	-3	-6	-7	-10	-12	-14

$p_{s,t}$	1	1	2	1	2	2
D_1	S	D_1	D_1	D_1	D_1	D_1
D_2	S	D_1, D_2	D_1	D_2	D_1, D_2	D_2

  Optimal log probability is obviously $-14 - 2 = -16$ (we need to include switching to the E state), so probability is $1/2^{16}$ (note that matrix a is log-matrix).

- We can observe two best paths from matrix p:
  - $S \to D_1 \to D_1 \to D_1 \to D_1 \to D_2 \to D_2 \to E$
  - $S \to D_1 \to D_1 \to D_2 \to D_2 \to D_2 \to D_2 \to E$

Question 5

This algorithm is just a slight modification of the Viterbi algorithm. We need to ensure that we go through state node q at step i.

First, we compute all the values of DP matrix up to step i using a standard Viterbi algorithm. Then, for the $i + 1^{st}$ step we compute the value for each state k using formula

$$DP\,[k]\,[i+1] = \log\left(e_k\left(x_{i+1}\right)\right) + DP\,[q]\,[i] + \log\left(a_{qk}\right)$$

and then we continue with standard Viterbi algorithm for steps $i + 2$, $i + 3$, $\ldots$, N.

Question 6

Probability of X_i being aligned with Y_j is the same as the probability of being in state m at step (i, j) and emitting symbols jointly. We already know that

$$P\left[\pi_{i,j} = M \mid X, Y\right] = \frac{f_M\left(i, j\right) \cdot b_M\left(i, j\right)}{P\left[X, Y\right]},$$

where f_M and b_M are forward and backward probabilities for ending in state M (match, alignment), and $P\left[X, Y\right]$ is the probability of generating sequences X and Y over all possible paths. We also know (lectures, book) how to compute all values of f_M and b_M, and $P\left[X, Y\right]$ in $O\left(n \cdot m\right)$ time by using the following two formulas:

$$f_q\left(i, j\right) = e_q\left(i, j\right) \cdot \left[f_X\left(i-1, j\right) \cdot a_{Xq} + f_Y\left(i, j-1\right) \cdot a_{Yq} + f_M\left(i-1, j-1\right) \cdot a_{Mq}\right]$$

$$b_q\left(i, j\right) = e_X\left(i+1, j\right) \cdot b_X\left(i+1, j\right) \cdot a_{qX} + e_Y\left(i, j+1\right) \cdot b_Y\left(i+1, j+1\right) \cdot a_{qY} + e_M\left(i+1, j+1\right) \cdot b_M\left(i+1, j+1\right) \cdot a_{qM}$$

In the end, desired answer is

$$P\left[X, Y\right] = \left[f_X\left(n, m\right) + f_Y\left(n, m\right) + f_M\left(n, m\right)\right] \cdot \tau,$$

where τ denotes the weight to the *end* node.